

# On a time-discrete approach to solving Navier-Stokes systems.

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**DNS.** This article advocate a new approximate scheme for Navier-Stokes systems which is described below: when we let  $\Omega$  and  $S$  be a domain in  $\mathbb{R}^N$  and it's boundary ( $N \in \mathbb{N}$ ),  $T$  a positive number and put  $a \in \overset{\circ}{W}_2^1(\Omega; \mathbb{R}^N)$  with  $\operatorname{div} a = 0$ , it is

$$\left\{ \begin{array}{l} \frac{\partial v}{\partial t} - \Delta v + (v \cdot D)v + \operatorname{grad} p = 0 \quad \text{in } \Omega \times (0, T), \\ \operatorname{div} v = 0, \\ v|_{\Omega \times \{0\}} = a \quad \text{and } v|_{S \times (0, T)} = 0 \end{array} \right. \quad (1)$$

for a function  $v : \Omega \times (0, T) \rightarrow \mathbb{R}^N$ .

The symbols employed here follows from a book of [1].

Our new scheme is to consider the following variational setting: Set  $N_T = [T/h]$  and  $n \in \{1, 2, \dots, N_T\}$ ,  $h > 0$  and put  $v_0 = a$ : Let us suppose a sequence of mappings  $\{v_k\}$  and of functionals  $\{I_k\}$  ( $k = 1, \dots, n-1$ ) be given. Then we adopt  $v_n$  ( $n = 1, 2, \dots, N_T$ ) as a minimizer of the functional

$$I_n[v] = \int_{\Omega} \left( \frac{|v(x) - v_{n-1}(x - hv_{n-1}(x))|^2}{2h} + \frac{1}{2} |Dv(x)|^2 \right) dx \quad (2)$$

in the class  $\overset{\circ}{W}_2^1(\Omega; \mathbb{R}^N) \oplus \overset{\circ}{\mathbf{J}}(\Omega)$ .

Since  $I[v]$  is convex and lower-semicontinuous, we can readily see that such a  $v_n$  exists in  $\overset{\circ}{W}_2^1(\Omega; \mathbb{R}^N) \oplus \overset{\circ}{\mathbf{J}}(\Omega)$ .

We hereafter call a sequence of  $\{v_n\}$  ( $1, 2, \dots, N_T$ ) *DNS* and note that  $v_n$  belongs to  $\overset{\circ}{W}_2^2(\Omega; \mathbb{R}^N)$  and satisfies the Euler-Lagrange equation

$$\frac{v_n(x) - v_{n-1}(x - hv_{n-1}(x))}{h} - \Delta v_n(x) \in \overset{\circ}{\mathbf{J}}^{\perp}(\Omega). \quad (3)$$

Reader should remark that the variational approach is not crucial to construct our *DNS* but important to discuss the nested term  $v_{n-1}(x - hv_{n-1}(x))$

inspired by the material derivative. From Helmholtz-decomposition lemma, we state that there exists a function  $p \in W_2^1(\Omega)$  such that

$$\frac{v_n(x) - v_{n-1}(x - hv_{n-1}(x))}{h} - \Delta v_n(x) = Dp(x), \quad (4)$$

that is equivalent to

$$\begin{aligned} & \frac{v_n(x) - v_{n-1}(x)}{h} + \int_0^1 v_{n-1}(x) \cdot Dv_{n-1}(x - h\tau v_{n-1}(x)) d\tau \\ & - \Delta v_n(x) = Dp(x) \end{aligned} \quad (5)$$

in the sense of distribution. This is a reason why we can regard (2) or (3) as an approximate formula for Navier-Stokes systems.

Throughout the paper we assume that

$$|Dv_n| = O(1/\sqrt{h}), \quad (A)$$

which seems quite natural because the differential coefficient of the first order with respect to *time* is as same as the twice *space*-differentials in the heat equation.

**Results.** We introduce the energy decay estimate directly obtained from (3):

Theorem 1: (The Energy Estimate). The following holds:

$$\begin{aligned} & \frac{h}{2} \sum_{n=1}^{N_T} \int_{\Omega} \left| \frac{v_n(x) - v_{n-1}(x - hv_{n-1}(x))}{h} \right|^2 dx + \frac{1}{2} \int_{\Omega} |Dv_n(x)|^2 dx \\ & \leq Ce^{CT} \int_{\Omega} |Da(x)|^2 dx \end{aligned} \quad (6)$$

for any positive integer  $n$  in  $\{1, 2, \dots, N_T\}$ , where  $C$  is a positive constant independent of  $h$ .

Proof of Theorem 1.

By substituting  $v_{n-1}(x - hv_{n-1}(x))$  as a comparative mapping for  $v$  in (2), extending it to 0 outside  $\Omega$ , computing the change of variables  $y = x - hv_{n-1}(x)$  with  $\text{div } v_n = 0$ , and using assumption on  $v_n$ : (A), we arrive at

$$\begin{aligned} & \int_{\Omega} \frac{|v_n(x) - v_{n-1}(x)|^2}{2h} dx + \int_{\Omega} |Dv_n(x)|^2 dx \\ & \leq (1 + Ch) \int_{\Omega} |Dv_{n-1}(x)|^2 dx, \end{aligned} \quad (7)$$

where  $C$  is a positive constant independent of  $h$ . The recursive usage above enjoys (6).  $\square$

In the below we prove the existence of a weak solution of Navier-Stokes systems: Before stating our theorem, we prepare a few symbols: Set  $t_n = nh$  ( $n = 1, 2, 3, \dots, N_T$ ) and

$$\begin{aligned} v_{\bar{h}}(t, x) &= v_n(x) \\ v_h(t, x) &= \frac{t - t_{n-1}}{h} v_n(x) + \frac{t_n - t}{h} v_{n-1}(x) \quad (t_{n-1} < t \leq t_n). \end{aligned}$$

When no ambiguity may arise, we say a pair of functions  $v_{\bar{h}}$  and  $v_h$  to be *DNS*;

**Theorem 2: (Main Theorem).** Under the hypothesis (A), *DNS* converge weakly-star to a function  $v$  in  $V_2(\Omega \times (0, T); \mathbb{R}^N)$  as  $h \searrow +0$ . Besides it converges strongly to  $v$  in  $L^2(\Omega \times (0, T))$ ;  $v$  is a weak solution of Navier-Stokes systems with

$$\int_0^T dt \int_{\Omega} \left\langle v(x, t), \frac{\partial \phi}{\partial t}(x, t) \right\rangle dx - \int_0^T dt \int_{\Omega} \langle Dv(x, t), D\phi(x, t) \rangle dx = 0 \quad (8)$$

for  $\forall \phi \in \dot{C}^\infty(\Omega \times (0, T))$  with  $\operatorname{div} \phi = 0$ .

Furthermore  $v$  satisfies

$$\int_{\Omega} |Dv(x, t)|^2 dx \leq Ce^{CT} \int_{\Omega} |Da(x)|^2 dx \quad (9)$$

for any time  $t$  in  $(0, T)$  where  $C$  is a positive universal constant.

Proof of Theorem 2.

The former is directly obtained by Theorem 1 combined with Rellich Kondrachev theorem and Poincaré inequality.

Next we claim that the convergent function  $v$  is actually a weak solution of Navier-Stokes systems; Since *DNS* implies

$$- \int_0^T dt \int_{\Omega} \left\langle v_h(x, t), \frac{\partial \phi}{\partial t}(x, t) \right\rangle dx + \int_0^T dt \int_{\Omega} \langle Dv_{\bar{h}}(x, t), D\phi(x, t) \rangle dx = 0 \quad (10)$$

for  $\forall \phi \in \dot{C}^\infty(\Omega \times (0, T))$  with  $\operatorname{div} \phi = 0$ ,

using results in the former, we can pass to the limit of  $h \searrow 0$  to verify the latter.

## References

- [1] Ladyžhenskaya, O. A., The Mathematical Theory of Viscous Incompressible Flow, (1963), Gordon and Breach.